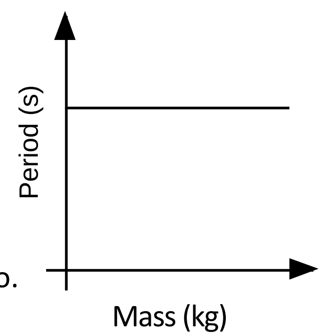
Graphical Analysis and Mathematical Models

Interpreting The Graph

The purpose of doing an experiment in science is to try to find out how nature behaves given certain constraints. In physics this often results in an attempt to try to find the relationship between two variables in a controlled experiment. Sometimes the trend of the data can be loosely determined by looking only at the raw data. The trend becomes more clear when one looks at the graph. In this course, most of the graphs we make will represent one of three basic relationships between the variables. These are 1) no relation 2) direct relation and 3) inverse relation. By examining the graph, you should be able to determine the type of relationship. To more specifically describe the relationship between the variables in an experiment, you will be expected to develop an equation. An equation which describes the behavior of a physical system (or any other system for that matter) can be called a **mathematical model**. The information which follows will describe each of the basic types of relationships we tend to see in physics. It is also the process for arriving at a mathematical model (equation) to more fully (and simply) describe the relationship between the variables and the behaviors of physical systems.

No Relation

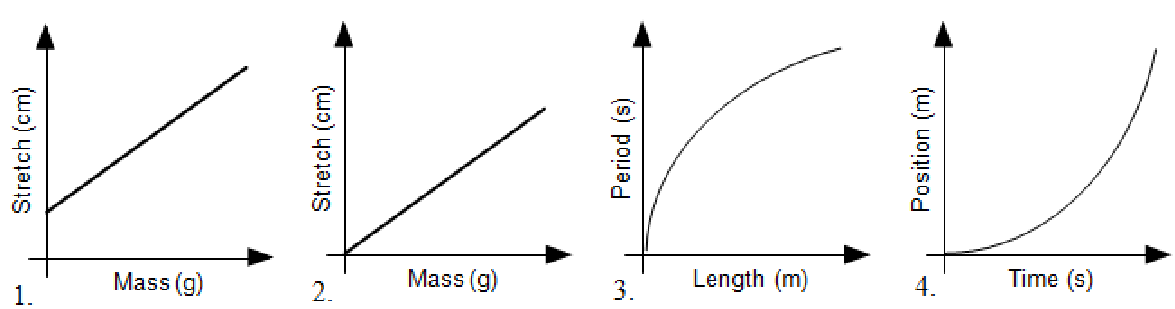


One possible outcome of an experiment is that changing the independent variable will have no effect on the dependent variable. When this happens we say that there is no relationship between the variables. The graph shown is an example of an experiment which demonstrates this lack of a relationship. As the independent variable increases, the dependent variable will stay the same, the resulting graph is a horizontal line. The slope of a horizontal line is always zero. Even though such a graph demonstrates no relationship between the variables, the equation of the line can still be determined.

Here is an example of the mathematical analysis of the graph for the Period vs. Mass graph shown above. Using the slope intercept form of a line (y = mx + b), we will substitute Period (P) in for the “y” variable, Mass (m) in for the “x” variable, solve for the slope (m) using the rise over run formula and substitute into the equation and lastly read the y-intercept (b) from the graph and substitute. The above Period vs Time graph shows a slope of zero and let’s assume the y-intercept is 5s. Substituting these values in would give us P = (0)m + 5s, simplified into a final mathematical equation: P = 5s.

Direct Relations

Another possible outcome of an experiment is that increasing the independent variable will cause the dependent variable to increase as well. We call this type of relationship a direct relation. The graph of a direct relation can take many forms, each of which represents a different type of direct relation. The following sketches show some different examples of direct relations.



Look at the examples above, notice that in each case, as the independent variable increases so does the dependent variable. Each one therefore qualifies as a direct relation. W e can also see, however, that the relations have significant differences.

In the first two cases the dependent variable changes by equal increments corresponding to equal changes in the independent variable. When a variable changes at a constant rate with respect to a second variable, the resulting graph will be a straight line and is said to be **linear.** Although examples 1 and 2 above both illustrate a **direct linear** relationship between the variables, only graph two is a **direct proportion.** A direct proportion must also pass through the origin. Often whether there is or is not a y-intercept is has significant meaning that will be addressed in a case by case situation. When comparing graphs 1 and 2 above, the y-intercept determines whether the spring started in a stretched or relaxed position. **The mathematical symbol for proportional is $\propto$.**

Graphs 3 and 4 are examples of direct relationships which are non-linear and are represented as curves. Graph 3 suggests that as Length increases, the Period increases as well, but by smaller and smaller increments. Equal increases in the x-variable are accompanied by decreasing increases in the y-variable. Graph 4, on the other hand, suggests that as Time increases, the Position increases by bigger and bigger increments. In other words, equal increases in the x-variable are accompanied by increasing increases in the y-variable. While many mathematical relationships could conceivably yield graphs similar to 3 and 4, we will find that in most of our experiments in this course, graphs that look like 3 and 4 are described by equations which are parabolic. For a parabola whose vertex is at the origin, like graph 3, the relation can be represented by an equation of the form $y^2 = kx$. Graph 4 might be represented by an equation of the form $y = kx^2$. When graphs like 3 and 4 are represented by equations in these forms, we will call them square relations.

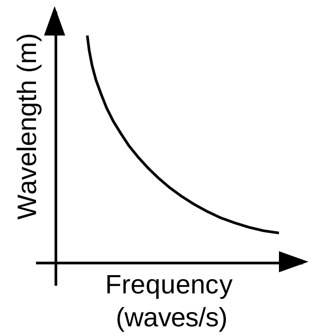
Let us now look at how to determine the specific mathematical models that are suggested by our various direct relationship graphs. When the data collected yields a linear graph we will proceed to writing the mathematical equation that describes the relationship between the variables. This is done by using the slope intercept form of the equation of a line (like the example earlier). With linear mathematical models we know many characteristics of the variables and can make predictions without actually doing further testing. When we are evaluating real data, we will need to decide whether or not the graph should go through the origin. Given the limitations of the experimental process, real data will rarely yield a line that goes perfectly through the origin. The first rule of order when trying to determine whether or not a direct linear relationship is indeed a direct proportion is to ask yourself what would happen to the dependent variable if the independent variable were zero. In many cases we can reason from the physical situation being investigated whether or not the graph should logically go through the origin. Sometimes, however, it might not be so obvious. In these cases we will employ a rule of thumb which is used only for this course and has no real statistical validity. It is called the 5% rule. It simply says that if the y-intercept does not exceed 5% of the maximum value of your y-axis data, we will assume that the y-intercept is negligible and will call it zero. If the y-intercept exceeds this arbitrarily chosen value of 5% of the maximum y data, we will assume that it has some physical significance and will go about trying to determine that significance.

Linearizing the Graph

When the data collected is non-linear and looks like it might be parabolic (see graphs 3 and 4 above), we will employ a powerful technique called **linearizing the graph** to determine whether or not the graph is truly a parabola with its vertex at the origin. Since these types of graphs are curved we cannot determine the equation that fits the data using the slope intercept equation because the slopes of these graphs are constantly changing. If the trend of the graph shows a parabola, like graph 4, and the graph bends away from the x-axis; it can be referred to as a “top opening graph”. To linearize this data you need to square the independent variables and generate a **test plot,** which is simply a new graph, of the y-variable vs. x-variable^2. This should produce a linear graph and we can now use this new linear graph to write a mathematical model using the slope intercept form equation. Note that the independent variable is now squared and should be represented as such in the equation $y = mx^2$. Graph 3 bends away from the y-axis and is referred to as a "side opening graph". To linearize this data we can take the square root of the independent variable and then generate a test plot of y-variables vs. the square root of the x-variables to produce the linear graph from which the mathematical equation is written. Note that the dependent variable could be squared and generate a y-variable squared vs x-variable graph that will be linear, but it is common practice to teach the manipulation of the independent variable when linearizing data.

Inverse Relations

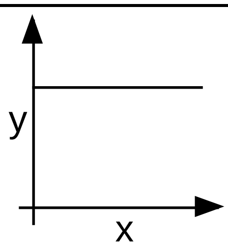
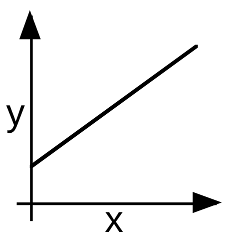
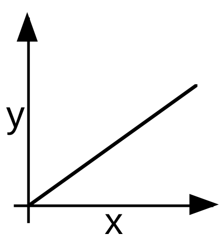
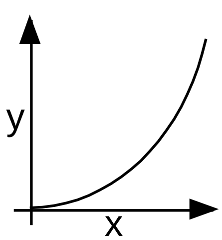
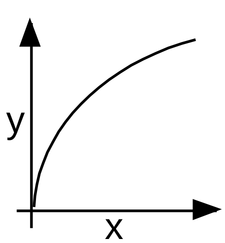
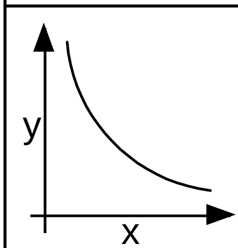
The final type of fundamental relationship that we will study is the **inverse relation**. An inverse relation is basically the opposite of a direct relation. In an inverse relation, as the independent variable increases the dependent variable decreases. This can take multiple forms, but the most common type that we will encounter in this physics course looks like the sketch to the right. To linearize this data we must take the inverse (or reciprocal) of the independent variables and generate a test plot from which to write our mathematical equation. (Note: the inverse of the dependent variables would also work, but again convention is to manipulate the independent variable.)



While the relations described above do not describe every possible physical situation that might be encountered, it does serve as the basis of a great many of them and will cover nearly every situation we are likely to encounter in an introductory physics course. Sometimes our test plot will not yield a linear relation initially, but it might suggest yet another test plot. In most situations in this course, a maximum of three graphs will allow us to linearize the data and obtain a mathematical equation. See the next page for a summary that will serve as a handy reference sheet for this class.

Summary--Mathematical Models from Graphs

One of the most effective tools for the visual evaluation of data is a graph. The ability to interpret what the graph means is an essential skill. You will be expected to learn to describe the relationship between the variables on a graph in two ways. One way will be to give a written statement of the general relationship between the two variables. The second is to develop an equation which will describe the relationship between these variables mathematically. We will call this equation a mathematical model of the physical relationship.

Graph Shape	Written Relationship	Modification Required to Linearize Graph	Mathematical Model
	***No Relation*** the same. There is no relationship between the variables.	***No Modification***	***$y = b$***
	***Linear Relation*** As x increases, y increases. y varies directly and linearly as x.	***No Modification Required***	***$y = mx + b$***
	***Direct Proportion*** As x increases, y increases proportionally. y is directly proportional to x.	***No Modification Required***	***$y \propto x$*** ***$y = mx$***
	***Parabolic Relation*** As x^2 increases, y increases proportionally. y is directly proportional to x^2. (square)	***Graph y vs x^2***	***$y \propto x^2$*** ***$y = mx^2$***
	***Parabolic Relation*** As x increases, y^2 increases proportionally. y^2 is directly proportional to x. (square root)	***Graph y^2 vs x*** ***Or*** ***Graph y vs $\sqrt{x}$***	***$y^2 \propto x$*** ***$y^2 = mx$***
	***Inverse Proportion*** As x increases, y decreases. y is inversely proportional to x.	***Graph y vs 1/x***	***$y \propto 1/x$*** ***$y = m(1/x)$***

Adapted from "Experimental Design and Graphical Analysis of Data" Rex P. Rice-2000